Microeconomics

A12749W1

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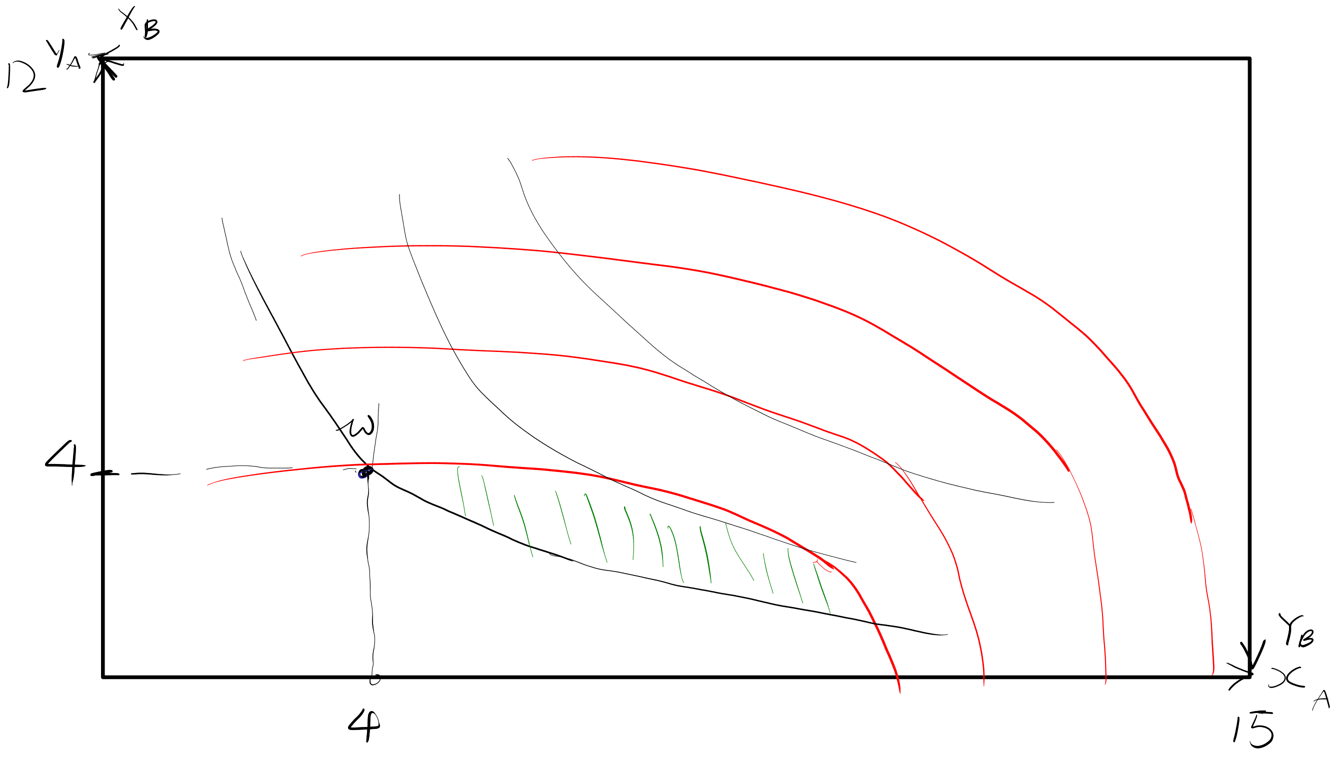
Part A

In Part A, the weight assigned to each question (as a percentage of the total for Part A) is indicated in square brackets

1. Consider an exchange economy with two consumers and two types of good. Consumer A has preferences represented by the utility function $u^A(x^A, y^A) = 2\ln x^A + \ln y^A$, and an endowment of $(4, 4)$ (that is, 4 units of each good); B has preferences represented by the utility function $u^B(x^B, y^B) = \ln x^B + 2\ln y^B$, with an endowment of $(11, 8)$ (that is, 11 units of good x and 8 units of good y).

   (a) Explain why the endowment is not Pareto efficient.
   Illustrate this situation using an Edgeworth box.



The endowment is PE if their MRSes are equal (slope of their indifference curves are tangent).

$$MRS_A = \frac{\frac{dU}{dx_A}}{\frac{dU}{dy_1}} = \frac{\frac{2}{x}}{\frac{1}{y}} \Rightarrow \frac{2y}{x}$$

$$MRS_B = \frac{\frac{dU}{dx_B}}{\frac{dU}{dY_B}} = \frac{\frac{1}{x}}{\frac{2}{y}} = \frac{1}{2}\frac{y}{x}.$$

At the initial endowment,

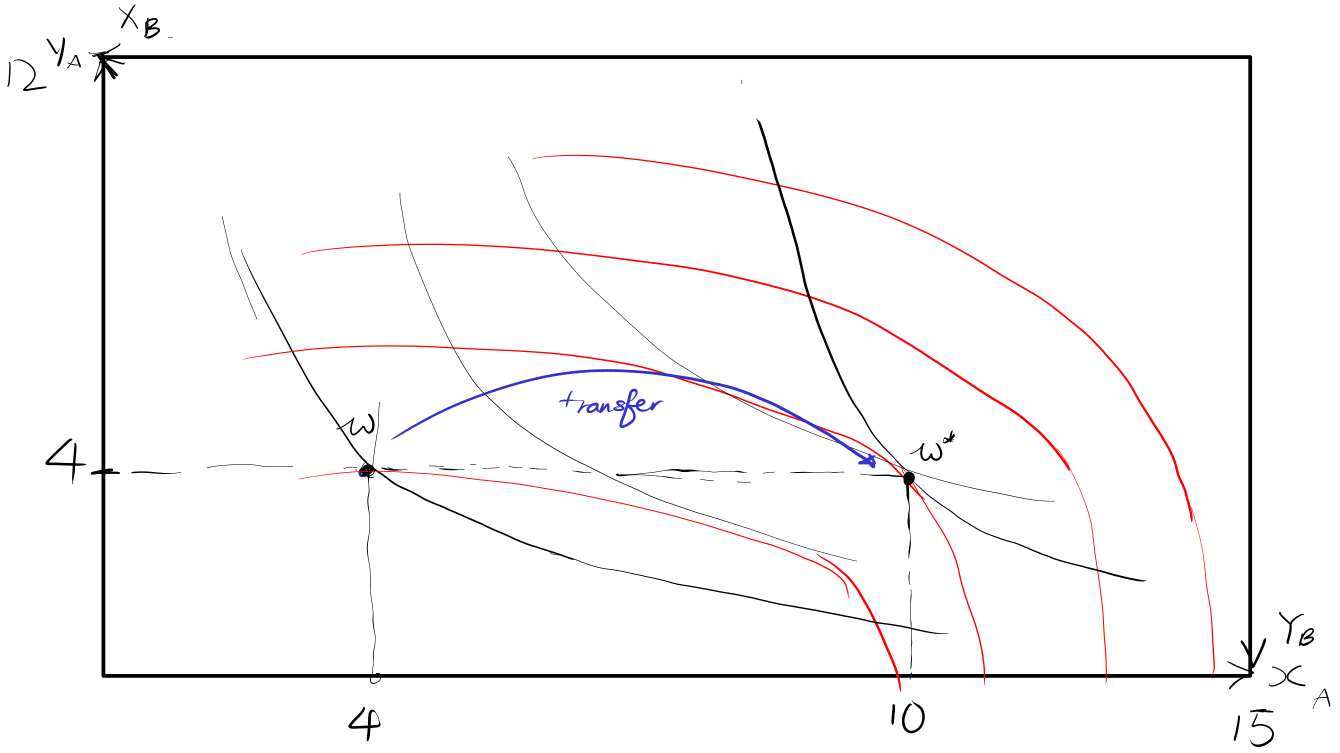
$$MRS_A = \frac{2(4)}{4} = 2$$

$$MRS_B = \frac{1}{2}\left(\frac{8}{11}\right) = \frac{4}{11}$$

Since the MRSes are not equal, this allocation is not PE.

We can see it in the Edgeworth Box: the MRS_A is relatively steep and MRS_B flat, and there will be an area (shaded in green) which Pareto-dominate the initial endowment.

(b) Suppose that good y cannot be transferred between the consumers.
Find the transfer (of good x) that would result in a Pareto efficient allocation.
Mark this allocation on your diagram.



Again, the PE must be at the point where (i) MRSes equal; and (ii) $Y_A = 4$
$Y_B = 8$.

$$MRS_A = \frac{2y}{x}, \qquad MRS_B = \frac{1}{2}\frac{y}{x}.$$

Subbing in Y_A and Y_B and demanding equality,

$$\frac{2(4)}{x_A} = \frac{1}{2}\frac{(8)}{x_B}$$

$$\frac{8}{x_A} = \frac{4}{x_B}$$

$2x_B = x_A$ Given that $X_A + X_B = 15$,

$x_A = 10$, $x_B = 5$.

(c) Can the initial allocation and the allocation in part (b) be ranked using the Pareto criterion? Explain why or why not.

No, it cannot. This is because consumer A is made better off but consumer B is made worse-off. For states to be rankable under the Pareto Criterion, each person needs to weakly prefer his allocation in x^* to x. But this isn't satisfied here.

(d) Now suppose that good y can also be transferred between the consumers. Show that the allocation that would be chosen by a social planner whose objective was to maximise the sum of the utilities of the two consumers is the same as the efficient allocation from part (b).
Is this different from the allocation that a social planner would choose if their objective was to maximise the minimum of the utilities of the two consumers? How do you know?

[25%]

The social planner's objective function is to maximise the sum of ulitities, i.e.

$$U = 2\ln(x_A) + \ln(Y_A) + \ln(15-X_A) + 2(12-Y_A) .$$

Here we simply differentiate to find the optimal quantities:

$$\frac{dU}{dX} = \frac{2}{X_A} + \frac{-1}{15-X_A} = 0 \qquad (1)$$

$$\frac{dU}{dY} = \frac{1}{Y_A} + \frac{-1}{24-2Y_A} = 0 \qquad (2)$$

Solving, (1)

$$\frac{2}{X_A} = \frac{1}{15 - X_A}$$

$$30 - 2X_A = X_A$$

$$\Rightarrow X_A = 10.$$

(2)

$$\frac{1}{Y_A} = \frac{1}{24 - 2Y_A}$$

$$24 - 2Y_A = Y_A$$

$$\Rightarrow Y_A = 8.$$

$\Rightarrow$ The allocation chosen by a utilitarian social planner $\{(10, 8), (5, 4)\}$, is exactly the PE allocation found in part (b). □

This allocation would be different if the social planner were Rawlsian. This is because

$$u(A) = 2\ln(10) + \ln(8) > u(B) = \ln(5) + 2\ln 4$$

and a Rawlsian social planner maximins; that is, $\max\{\min(u_A, u_B)\}$. It can be shown that it's always better to $u_A = u_B$ with a Rawlsian social planner.

2. Two colleges, Larches and Osier, are currently equal in the academic league table, but each is considering extra coaching for their students. They simultaneously decide whether to *Maintain* current levels, *M*, or *Increase* the level of coaching, *I*, payoffs being given by:

		LARCHES	
		Maintain	*Increase*
OSIER	*Maintain*	0 0	2 −1
	Increase	−1 2	−4 −4

(a) Define the Nash equilibrium of a game.

Find all the Nash equilibria of the one-shot game above (in both pure strategies and mixed strategies, if any).

Calculate the expected payoffs of each college in each of the equilibria.

The Nash equilibrium is a strategy profile such that no player can obtain a higher pay off by employing a different strategy profile s_i' than his strategy in the NE. (No profitable unilateral deviations)

		M p	I $1-p$
π	M	0 , 0	$\underline{-1}$, $\underline{2}$
$(1-\pi)$	I	$\underline{2}$, $\underline{-1}$	−4 , −4

This is the Hawk-Dove game.

Pure strategies:

$\{M, I\}$, $\{I, M\}$.

Mixed strategy: a player must be indifferent between all pure strategies in his support. Hence let p be the probability of Larches playing M. ~~Oriel~~ Osier is indifferent when:

$$\cancel{p(0)} + (1-p)(-1) = p(2) + (1-p)(-4)$$

Solving, we obtain

$$-1 + p = 2p - 4 + 4p$$

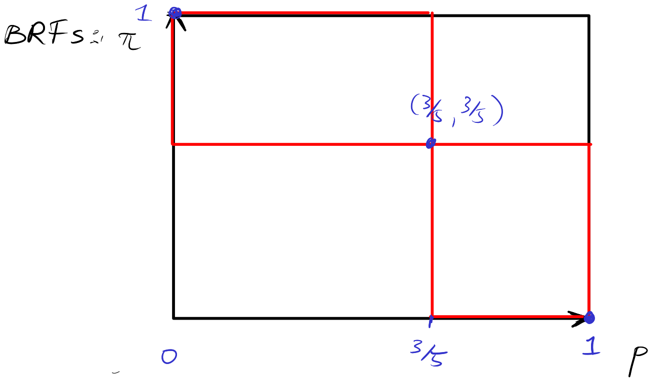
$$p - 1 = 6p - 4$$

$$p = 3/5 .$$

The game is symmetric, so $\pi = 3/5$ as well.

There is thus a mixed NE where

$$\left\{ \begin{Bmatrix} \text{Maintain} : 3/5 \\ \text{Increase} \;\; 2/5 \end{Bmatrix} , \begin{Bmatrix} '' & : 3/5 \\ I & : 2/5 \end{Bmatrix} \right\}.$$



Payoffs:

In $\{M, I\}$ and $\{I, M\}$

payoffs are $(2, -1)$ and $(-1, 2)$

In the mixed NE payoffs are

$$3/5 \cdot 3/5\,(0) + 3/5 \cdot 2/5 \cdot (-1)$$
$$+\ 3/5 \cdot 2/5\,(2) + 2/5 \cdot 2/5\,(-4)$$

$$\Rightarrow (-10/5, -10/5)$$

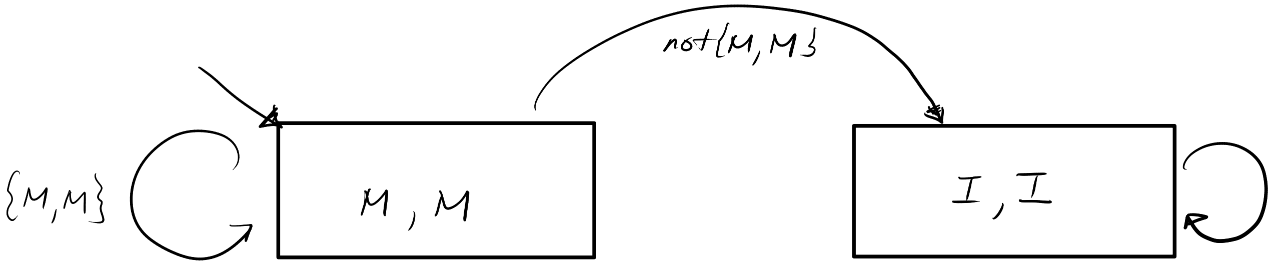
(b) Explain what is meant by a *subgame perfect equilibrium* of a repeated game.

(c) Suppose that the two colleges repeat the above stage game infinitely often. Show that (*Maintain*, *Maintain*) in every period can be sustained as the outcome of a subgame perfect equilibrium for a discount factor sufficiently close to one. [*Hint*: consider a grim trigger strategy with the threat of reverting to a suitable Nash equilibrium of the stage game from part (a).]

[25%]

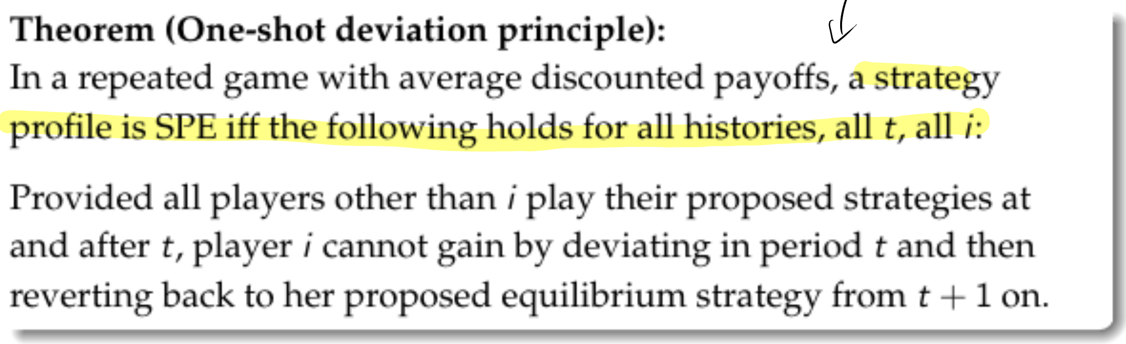
b) A subgame-perfect equilibrium is a strategy profile that is a Nash equilibrium in every subgame.

c) Here's the FSM for the grim trigger SPE:



That is, players start off playing $\{M, M\}$ and play $\{I, I\}$ forever once any player plays anything other than $\{M, M\}$.

from Peter Eso's slides



Theorem (One-shot deviation principle):
In a repeated game with average discounted payoffs, a strategy profile is SPE iff the following holds for all histories, all *t*, all *i*:

Provided all players other than *i* play their proposed strategies at and after *t*, player *i* cannot gain by deviating in period *t* and then reverting back to her proposed equilibrium strategy from $t + 1$ on.

By the one-shot deviation principle,
We can check that the grim trigger strategy can be sustained as SPE by checking one-shot deviations only from every state of the FSM.

Either player

1. Consider state $\boxed{M, M}$ of the automata :
has a profitable one-shot deviaton iff

$$\overset{\text{cooperate}}{0} < \underset{\text{one-shot deviation}}{2} \overbrace{- 4\delta + -4\delta^2 + -4\delta^3 + \cdots}^{\text{grim trigger}}$$

When $\delta \to 1$, that's definitely the case:

$$4\left(\sum_{n=1}^{\infty} \delta\right) \longrightarrow \frac{4}{1-\delta} \longrightarrow \infty \text{ when } \delta \to 1$$

2. In state $\boxed{I, I}$, there is no incentive to deviate.

Hence, the grim trigger strategy is a SPE when $\delta \to 1$.

3. (a) Define the certainty equivalent of a lottery, and explain how it can be used to determine whether or not someone should participate in a given lottery.

Consider a lottery L that gives wealth in either (w_1, w_2) with prob $(p, 1-p)$. Then the certainty equivalent is the amount of wealth w^* that would give the same amount of utility as the lottery:

$$w^* \text{ s.t. } u(w^*) = p(u(w_1)) + (1-p)(u(\omega_2))$$

A person should participate in a given lottery if the certainty equivalent of that lottery exceeds his current weath.

Janet is an expected utility maximiser who prefers more to less, and whose broad attitude to risk (risk averse, risk neutral, or risk loving) is independent of her wealth.

There are two projects. If Janet invests in project X, her final wealth will be either 6 or 30, each equally likely; if she invests in project Y, her final wealth will be either 12 or 20, again each equally likely. She is prepared to invest in one or the other, but is not allowed to invest in both.

(b) She tells her friend Sam that she prefers Y to X, from which Sam infers that Janet is risk averse, reasoning as follows:

> Consider project Z which would leave Janet with either 14 or 22, each equally likely. She would certainly prefer Z to Y, and if she were risk neutral or risk loving then she would be content with X instead of Z, and consequently prefer X to Y – but she doesn't, so she must be risk averse.

Evaluate Sam's argument.

$X : \{6, 30\}$

$Y = \{12, 20\}$

$Z : \{14, 22\}$

Sam's argument makes sense. Z is preferred to Y for all EU maximisers: statewise dominance: the low outcome is higher and the high outcome is higher.

Now comparing Z and X. Their variances are

$$Var(X) = \frac{1}{2}[(6-18)^2 + (30-18)^2] = 144$$

$$Var(Z) = \frac{1}{2}[(14-18)^2 + (22-18)^2] = 16$$

Additionally, the lotteries have the same expected wealth: $\mathbb{E}(Z) = \mathbb{E}(X) = 18$.

X is similar to a mean-preserving spread of Z. Here, Since the expected wealth is equal, any risk neutral agent would be indifferent, and any risk-loving agent would prefer X to Z. as the variance of X is Larger than Y. that is,

Risk Neutral: $X \sim Z \succ Y$

Risk Loving: $X \succ Z \succ Y$

and if Janet prefers Y, she must be RA.

(c) Janet now hears that the cost of investing in project Y is going to increase, effectively reducing her final wealth. If her utility from final wealth w is $\ln w$, what is the largest cost increase Janet would find acceptable, above which she would prefer to invest in project X?

[25%]

Expected utility from X = $\frac{1}{2}\ln(6) + \frac{1}{2}\ln 30$

" " from Y = $\frac{1}{2}\ln(12-x) + \frac{1}{2}\ln(20-x)$

We are finding x where Janet is indifferent b/w both lotteries:

$$\frac{1}{2}\ln(12-x) + \frac{1}{2}\ln(20-x) = \frac{1}{2}\ln(6) + \frac{1}{2}\ln 30$$

We could solve it directly or compare certainty equivalents. Here I elect to solve directly.

$$\Rightarrow \ln(12-x) + \ln(20-x) = \ln(6) + \ln(30)$$

$$\Rightarrow e^{\ln(12-x)} e^{\ln(20-x)} = e^{\ln 6} e^{\ln 30}$$

$$\Rightarrow (12-x)(20-x) = 180$$

$$\Rightarrow x = 2. \square$$

4. There are two types of worker: H-types with high productivity θ_H, and L-types with low productivity θ_L; each worker knows their own type, but they have no outside option. Many risk-neutral firms compete for the services of the workers, but they cannot observe a worker's type.

   (a) Why would some of the workers like to be able to signal their type to the firms? What condition(s) would make a signal credible?

Without signalling, companies pay the average productivity $\lambda\theta_H + (1-\lambda)\theta_L$ to workers.

If possible (signal cost not too high), H-types want to signal so as to obtain their productivity θ_H in a competitive labour market. They would then get

$$\theta_H - \text{signal cost for H-type} \geqslant \lambda\theta_H + (1-\lambda)\theta_L .$$

A signal is credible if it is more costly for L-types than H-types to obtain, and it must be so costly for L-types that they would be better off not signalling;

$$\theta_L > \theta_H - \text{signal cost for L-types} .$$

(Note that signals don't have to be costly for H-types!)

Suppose that $\theta_H = 500$, $\theta_L = 400$, and that 3/5 of the workers are H-types, so 2/5 of the workers are L-types; the firms are aware of these productivity levels and proportions.

(b) When no signal is available, what is the market equilibrium outcome?

(c) Suppose the workers can acquire an education, observable by the firms – this would cost 80 for an L-type, but only 60 for an H-type.

Consider an outcome in which H-types signal and L-types don't, and the firms pay appropriate wages.
Is this outcome a market equilibrium? Explain why or why not.
If not, find an alternative outcome, and show that it is an equilibrium.
Is the equilibrium efficient?

[25%]

(b) Firms pay θ_P, the average productivity of the workers, which is

$$\theta_P = \lambda(\theta_H) + (1-\lambda)\theta_L$$
$$= \tfrac{3}{5}(500) + \tfrac{2}{5}(400)$$
$$= 300 + 160$$
$$= 460.$$

(c) Consider the separating equilibrium:

$$\theta_H = 500$$
$$\theta_L = 400,$$

(as firms must make 0 profits in a competitive labour market i.e. paying each worker their MPL,)

and only H-types signal.

This cannot be a CE, as L-types have an incentive to deviate and obtain $500 \div 80 = 420 > 400$

$$\theta_L < \theta_H + C_L$$
$$400 < 500 - 80$$
$$400 < 420$$

The alternative outcome is for both types not to signal and for firms to pay the pooled wage Θ_P.

Check: No profitable deviations for H-Types

H-types:

$$\Theta_P > \Theta_H - C_H$$

$$460 > 500 - 60$$

The equilibrium here is efficient.

10. Consider a duopoly in which two firms produce goods that are perfect substitutes, and they compete for one period only, simultaneously choosing their production level. The cost to firm j of producing a quantity $q_j \geq 0$ is $15q_j$, for $j = 1, 2$, and the inverse demand function in the market is

$$p(q) = 75 - q \quad \text{where } q = q_1 + q_2.$$

(a) Find the pure-strategy Nash equilibrium of this game.
What is the resulting price, and how much profit does each firm make? [20%]

To find the pure-strategy NE, we find the set of mutual BRs: where both firms maximise π_i:

$$\pi_1 = pq_1 - 15q_1$$

$$= (75 - q_1 - q_2)q_1 - 15q_1$$

Maximising profits,

$$\frac{\delta \Pi_1}{\delta q_1} = 75 - 2q_1 - q_2 - 15 = 0$$

$$\Rightarrow \frac{60 - q_2}{2} = q_1$$

Check SOC:

$$\frac{\delta' \Pi_{1}}{\delta' q_1} = -2q_1 < 0$$. So this is a maximum.

By symmetry,

$$q_2 = \frac{60 - q_1}{2}$$ □

These are each firm's best responses to the other firm's quantity. Substituting q_2 into q_1 gives

$$\frac{60 - \left(\frac{60 - q_1}{2}\right)}{2} = q_1$$

$$30 - 15 + \frac{q_1}{4} = q_1$$

$$15 = \frac{3q_1}{4}$$

$$q_1 = 20.$$

⇒ by symmetry, $q_1 = q_2 = 20$.

⇒ The pure NE is $\{q_1 = 20, \; q_2 = 20\}$.

Prices and Profits

$$p = 75 - q_1 - q_2$$
$$= 35.$$

$$\pi_j = p(q_j) - 15q_j$$
$$= 35(20) - 15(20)$$
$$= 400.$$

⇒ Prices are £35, each firm makes £400.

Suppose that before the two firms play the above quantity-setting game, firm 1 can make a cost-reducing investment that has a lump-sum price of $K > 0$ and results in its cost function becoming $9q_1$. Firm 2 knows whether or not firm 1 made the investment before the firms choose how much to produce.

(b) If firm 1 decides to invest, what will be the Nash equilibrium of the quantity-setting game?
What is the resulting price, and how much profit does each firm make (ignoring the prior investment cost of firm 1)?
What are the reasons why firm 1 earns more than in part (a) while firm 2 earns less? [30%]

b) If firm 1 invests, its profit function becomes

$$\pi_1 = pq_1 - 9q_1$$

$$= (75 - q_1 - q_2)q_1 - 9q_1$$

Maximising profits,

$$\frac{\delta \pi_1}{\delta q_1} = 75 - 2q_1 - q_2 - 9 = 0$$

$$\Rightarrow \frac{66 - q_2}{2} = q_1$$

Here q_2's profit function remains unchanged, and so its BRF is still $q_2 = \frac{60 - q_1}{2}$.

Substituting q_2 gives

$$q_1 = \frac{66 - \frac{60 - q_1}{2}}{2}$$

$$= \frac{36 + \frac{q_1}{2}}{2}$$

$$= 18 + \frac{q_1}{4}$$

$$\Rightarrow q_1 = 18 \cdot 4/3 = 24, \text{ and } q_2 = \frac{60 - 24}{2} = 18.$$

The price of the goods is now

$$\begin{aligned} p &= 75 - 18 - 24 \\ &= 33 \end{aligned}$$

and profits are now

$$\begin{aligned} \pi_1 &= 33(24) - 9(24) \\ &= 24^2 \\ &= 576 \end{aligned}$$

$$\begin{aligned} \pi_2 &= 33(18) - 15(18) \\ &= 18^2 \\ &= 324 . \end{aligned}$$

Firm 1 is making more profit for two reasons:

1. its marginal cost of production has gone down due to investment;

2. Because goods are strategic substitutes in Cournot competition (the BRFs decrease in your opponent's quantity), the fact that Firm 1 produces more means that ~~F~~ Firm 2 backs down, resulting in more profit (by increased Q and not-as-low prices) accruing to firm 1.

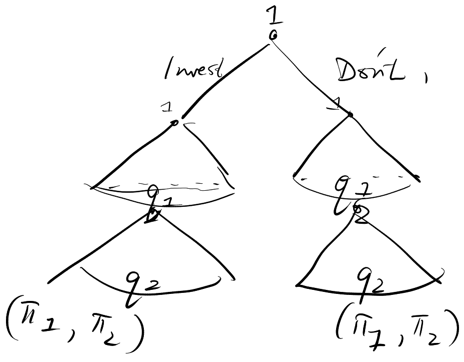
Similarly, Firm 2 makes less profit because it has to back down by producing a smaller quantity of goods.

(c) Consider the two-stage game in which firm 1 chooses whether to invest or not in the first stage, and, having noted the investment decision of firm 1, the firms engage in quantity competition in the second stage.

In a subgame-perfect Nash equilibrium of the above two-stage game, what is the largest value of K that would result in firm 1 making the investment?

When K is low enough for firm 1 to make the investment, describe fully the strategies of the two firms in the subgame-perfect Nash equilibrium. [25%]

We can draw the game tree as follows:



Firm 1 invests only if cost is low enough such that the profits from investing less the cost are greater than not investing:

$$\underset{\substack{\uparrow \\ \text{profit if not invest}}}{400} < \underset{\substack{\uparrow \\ \text{profit if invest}}}{576} - K$$

$$K < 176.$$

When K is low enough so that firm 1 would like to invest, the SPE is as follows:

Firm 1.
Stage 1:
Invest
Stage 2:
If invested in stage 1, produce $q = 24$.
Otherwise, produce $q = 20$.

Firm 2
Stage 2
If Firm 1 invested, produce $q = 18$
Otherwise, produce $q = 20$

The work we've done in part a) and b) tell us this is an SPE. When the firm invests, we are in the subgame solved in part b), and we found the NE there was $\{24, 18\}$. When it doesn't invest, we are in the subgame solved in part a), and the NE there was $\{20, 20\}$. Hence, we've verified that this profile is a~~n~~ NE in every subgame. Thus, it must be a SPE.

(d) Now suppose $K = 150$, but that before the game starts, firm 2 announces that at the second stage it will ignore firm 1's investment decision and produce its quantity from the equilibrium in part (a).

If firm 1 believed the announcement, would it invest?

Should firm 1 invest – that is, is firm 2's announcement credible? Explain. [25%]

No, it wouldn't. If firm 2 produces 20, then firm 1's best response is

$$q_1^* = \frac{66 - q_2}{2} = 23$$

and its profit would be

$$\pi_1^* = (75 - q_1^* - q_2^*)q_1^* - 9q_1^*$$

$$= (75 - 20 - 23)(23) - 9(23)$$

$$= (23)(23)$$

$$= 529.$$

But $529 - 150 < 400$, that is, the profit accruing to the firm if ~~he~~ it invests is less than the profit of the firm doesn't invest. And hence firm 1 will not invest. The NE, if the threat were credible, is

$$\left\{ \left\{ \begin{array}{l} \text{Don't Invest,} \\ q_1 = 23 \text{ or } q_1 = 20 \end{array} \right\}, \left\{ q_2 = 20 \right\}. \right\}$$

But this is not a credible threat and so firm 1 should just go ahead with the investment. Why not?

Because this NE isn't an SPE. In the subgame where firm 1 invests, firm 2's best response is to produce $q_2 = 18$ as producing $q_2 = 20$ would actually reduce its profits compared to producing $q_2 = 18$:

$$\underset{(\pi_2 \mid q_2 = 20)}{\underset{\uparrow}{(75 - 20 - 23)(20) - 15(20)}} = 160 < \underset{(\pi_2 \mid q_2 = 18)}{\underset{\uparrow}{326}}$$

So it would in fact choose $q_2 = 18$ were firm 1 to invest.

Thus, the threat isn't credible and Firm 1 should invest.

9. Explain the term “agency cost” in the context of Principal-Agent problems. Why does it arise? There are a number of situations under which the agency cost is absent (or zero). What are these situations? For each one, describe the outcome and how the Principal achieves it.

Contents

Introduction

Agency cost is the risk premium associated with the contract given to the agent by the principal to exert a high effort level. The risk premium associated with a lottery is the difference between the expected value of the outcome, and the certainty equivalent—which in the context of the principal-agent problem is simply the agent's reservation utility plus effort costs.

Broadly speaking, agency cost arises because effort is unobservable and agents are risk-averse. Because effort is unobservable, the principal has to pay an agent a high wage when the high outcome obtains, and low wage when the low outcome obtains to incentivise the agent to put in high effort. But because the principal now has to pay what is essentially a lottery—and the agent is risk-averse—the principal will have to pay a higher amount than the certainty equivalent in order to satisfy the agent's participation constraint. In this essay, I will set up the basic principal-agent model, then analyse all the possible situations, and show that agency cost only arises when the principal wants to induce high effort in both the observable and unobservable effort case.

Basic model setup

Suppose a risk-neutral principal derives a payoff

$$\Pi = y(e) - w(e)$$

which depends on output and the wage it pays to the agent. The output (and possibly wage) in turn depend on the effort exerted by the agent.

For ease of exposition, we assume that effort can only take low or high levels: that is, $e \in 0, 1$.

Let the probability of a "high" outcome y_H conditional on the agent exerting effort level 1 be denoted as $p(1)$, *mutatis mutandis* for effort level 0. That is, $p(1) = P(y = y_H | e = 1)$ and $p(0) = P(y = y_H | e = 0)$. Additionally, we demand that $p(1) > p(0)$: if the agent exerts high effort, the probability of a "high" outcome increases.

The principal wants to maximise his expected profit by choosing some wage contract (w^L, w^H) which induces the effort level e that maximises the following profit function:

$$\Pi = (1 - p(e))(y^L - w^L) + p(e)(y^H - w^H)$$

Finally, the agent's utility function is some function of his wage minus his effort cost (a function increasing in his effort).

$$U(w, e) = f(w) - e(e).$$

The agent also has a reservation utility $\bar{u}$, below which he will decline to work for the principal.

Model analysis

Now that we have set up the model, we can analyse the different cases where agency cost may be zero or absent. I will first analyse the base case, where effort is observable, and then see if there is any difference in the contracts paid to the agents in the following cases:

- Any agent when effort can be deduced from outcome
- Risk neutral agent
- Risk-averse agent when effort is unobservable:
  1. When the principal always prefers low effort
  2. When the principal prefers high effort when effort is observable but low effort otherwise
  3. When the principal always prefers high effort

Observable effort

When effort is observable, the principal offers a wage schedule depending on whether profit is maximised when the worker exerts low effort or high effort. This in turn depends on the agent's effort costs.

The agent has both a *participation* and an *incentive* constraint. The participation constraint ensures that the agent's reservation utility must be lower than his wage when he exerts either high or low effort, i.e. it is better to work than not at all. The incentive constraint is that $u(1) > u(0)$, i.e. it's better to exert high effort than low effort.

When effort is observable, the incentive constraint is irrelevant. The principal can simply make wage conditional on effort, and will set

$$(w|e) = \bar{u} + e(e),$$

that is, the wage conditional on effort level just equals the agent's reservation utility plus the effort incurred. This is the *participation constraint*: the agent (weakly) prefers to work for the principal. The principal can additionally set the wage conditional on its undesired effort to equal zero (it can really be any amount as long as it gives the agent a lower payoff).

The principal prefers to induce high effort if and only if the expected payoff from inducing high effort is greater:

$$p(1)y^H + (1 - p(1))y^L - w(1) > p(0)y^H + (1 - p(0))y^L - w(0),$$

where $w(\cdot)$ is the wage paid to the agent if he exerts a particular effort level. We have thus that

$$\boxed{w(i) = \bar{u} + e(i)}$$

in the case where effort is observable.

Unobservable effort

As mentioned, agency costs start to arise when effort is observable, but not always. I now examine the cases.

There is no agency cost when the outcomes are a sure thing so that effort is essentially observable

It's a bit pedantic, but let's mention the corner case for completeness. If $Pr[Y = 1|e = 1] = 1$ and $Pr[Y = 1|e = 0] = 0$ then while effort is not *technically* observable it might as well be, because effort corresponds perfectly to the outcome. In this case, agency cost is zero for the simple reason that there is no risk in the outcome.

There is no agency cost with a risk-neutral agent

Suppose the agent were risk-neutral. Without (much) loss of generality we can write his utility function as the following:

$$U(w, e) = w - e(e).$$

Suppose the principal wished to induce high effort. Then the principal must offer a wage schedule that satisfies the following participation and incentive constraints:

$$IC : p(1)w^H + (1 - p(1))w^L - e(1) \geq p(0)w^H + (1 - p(0))w^L - e(0),$$

or rewriting in terms of expectations,

$$\boxed{IC : \mathbb{E}[w|e = 1] \geq \mathbb{E}[w|e = 0] + e(1) - e(0).}$$

And because the risk-neutral agent's utility function is linear in wage, the PCs can be written as:

$$PC_1 : p(1)w^H + (1 - p(1))w^L - e(1) = \bar{u},$$

or (if the principal prefers low effort)

$$PC_0 : p(0)w^H + (1 - p(0))w^L - e(0) = \bar{u}.$$

The constraints all bind with equality because the agent is profit-maximising.

But now notice that we can rewrite the wage terms as the expected wage, and rearrange the participation constraints to give

$$\boxed{PC_i : \mathbb{E}[w|e = i] = \bar{u} + e(i)}$$

Compare this equation with the wage paid in the observable effort case. This equation tells us that the wage schedule can be *anything* as long as:

1. The expected wage when the agent exerts effort level i is equal to his reservation utility plus his effort cost, and
2. The expected wage when the agent exerts high effort is (weakly) greater than the expected wage when the agent exerts low effort plus the difference in effort costs.

Why does this result hold? Simply because the agent is risk-neutral, which means that the agent is indifferent between all lotteries with the same expected value. So the risk premium is zero here. And because the principal pays an expected wage equal to $\bar{u} + e(1)$, *exactly* the same as that in the observable effort case, the agency cost is zero.

There is no agency cost when principal always prefers to elicit low effort

I have defined agency cost (following the notes) as the risk premium associated with the contract given to the agent by the principal to exert a *high effort level*. By definition, therefore, agency cost cannot exist when the principal wishes to elicit low effort. But let us examine why agency cost cannot exist in this case.

If the principal wants to induce low effort, there is no change from the unobservable case. The principal can pay a fixed wage $w = w^H = w^L$ such that the participation constraint for low effort is satisfied with equality:

$$p(0)u(w^H) + (1 - p(0))u(w^L) - e_0 = \bar{u}$$

With $w = w^H = w^L$, this simplifies to

$$u(w) - e(0) = \bar{u},$$

which is once again exactly the PC in the case with observable effort, and hence there is no agency cost. Note that because $e(1) > e(0)$, if the PC for low effort is satisfied with equality, the PC for high effort will never be satisfied. So the principal is guaranteed to get low effort.

There is no agency cost (but a welfare loss) when the principal prefers high-effort in the observable effort case but low effort otherwise

As we've just shown, if the principal wishes to exert low effort when effort is unobservable, then agency cost does not exist by definition. But there is a welfare loss. If the principal wants to induce H in the observable case but L in the unobservable one, then it must be the case that the agency cost is greater than the increased expected surplus from inducing high effort. That must mean the principal is worse off.

There is agency cost when principal always prefers to elicit high effort

Finally, let us look at when agency costs arise.

Recall that the IC can be written as

$$p(1)u(w^H) + (1 - p(1))u(w^L) - e(1) \geq p(0)u(w^H) + (1 - p(0))u(w^L) - e(0).$$

That is to say, the agent will prefer to exert high effort to low effort when the expected utility from high effort exceeds the expected utility from low effort.

This means that the difference between the low wage and the high wage must be high. To induce high effort, the principal must offer a low wage that is low enough to satisfy the IC. We rearrange the incentive constraint and take the inverse to obtain

$$w^L \leq u^{-1}(u(w^H) - \frac{1}{p(1) - p(0)})$$

As usual, it must also offer a wage high enough to satisfy the participation constraint:

$$p(1)u(w^H) + (1 - p(1))u(w^L) \geq \bar{u} + e(1).$$

in other words, that the certainty equivalent of the wage when exerting high effort must give the agent at least its reservation utility. Again, we can rewrite this in terms of the expected utility:

$$\boxed{\mathbb{E}[u(w)|e = 1] \geq \bar{u} + e(1).}$$

Here Jensen's inequality bites. Because the agent is risk-averse, his utility function has diminishing marginal utility to wealth: that is to say,

$$\mathbb{E}[u(w)|e = 1] \geq u(\mathbb{E}[w|e = 1]).$$

What does this inequality imply? The principal can no longer pay a wage such that the expected wage $\mathbb{E}[w|e = 1]$ equals the reservation wage plus the effort cost, like in the observable effort and risk-neutral agent case! The expected wage the principal must pay is higher such that the *certainty equivalent* equals the reservation utility (plus effort cost). Thus, the agency cost is the difference between the expected wage in this unobservable contract, and the expected wage in the observable contract:

Why is this the case? When effort is not observable the incentive constraint means that the low wage must be low enough. But because the agent is risk-averse, this is not appealing to the agent. Hence the high wage must be high enough—and in particular high enough such that the certainty equivalent just satisifies the participation constraint. This is why the principal is worse off: it must pay a low wage low enough to incentivise high effort, but it must raise wages above to compensate the agent's risk aversion as a result.